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The representation space of the nuclear symplectic Sp(6, R) shell model

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Abstract. The problem of obtaining the $\mathrm{Sp}(6, R)$ irreducible representations that arise from the symmetrized products of A copies of the fundamental representation of $\mathrm{Sp}(6, R)$ is considered. A simple method is developed that rapidly leads to the determination of the $\mathrm{Sp}(6, R)$ content up to a prescribed cut-off. In practical calculations A is identified with the nucleon number. It is shown that for a given excitation energy cut-off, $n\hbar\omega$, and sufficiently large A, the number and multiplicities of the $\mathrm{Sp}(6, R)$ irreducible representations is fixed and the content can be expressed in an A-independent manner. Complete results are given for $n \leqslant 8$ and all $A \geqslant 16$. There is no difficulty in principle in going to higher excitations and details are given for determining the $\mathrm{Sp}(6, R)$ content for $A \leqslant 16$.

1. Introduction

Central to the nuclear symplectic shell model (Rowe 1985, Carvalho *et al* 1986) is the necessity to define the relevant $\mathrm{Sp}(6, R)$ representation space for an A-nucleon system where the number of nucleons, A, may be quite large, say ~ 100 or so. Even for A as small as 4, as in the case of the ^{4}He nucleus, it can be a major task to determine the collective subspaces that carry $\mathrm{Sp}(6, R)$ irreducible representations (Carvalho 1990). The basic problem is to obtain the $\mathrm{Sp}(6, R)$ irreducible representations that arise from the symmetrized products of A copies of the fundamental representation of $\mathrm{Sp}(6, R)$. Since the non-trivial unitary irreducible representations of the harmonic series of $\mathrm{Sp}(6, R)$ are necessarily of infinite dimension the number of irreducible representations contained in a symmetrized product is itself infinite. In practical calculations interest is usually restricted to a finite excitation energy $n\hbar\omega$ and hence it is desired to know the $\mathrm{Sp}(6, R)$ content up to a prescribed cut-off and hence of a finite number of $\mathrm{Sp}(6, R)$ irreducible representations. It is of some interest to know how the $\mathrm{Sp}(6, R)$ representation space depends on the nucleon number A if a given cut-off is assumed.

It is one of our key results in this paper to show that for a given cut-off, and sufficiently large A, the number of $\mathrm{Sp}(6, R)$ irreducible representations is fixed and their content can be expressed in an A-independent manner. Complete results are given for excitations up to a level $8\hbar\omega$ and all $A \geqslant 16$. There is no difficulty, at least in principle, in going to higher excitations and we specify how to obtain the $\mathrm{Sp}(6, R)$ irreducible representation content for $A \leqslant 16$.

Carvalho (1990) has already made a preliminary study of this problem making extensive use of Schur functions (S-functions) and S-function plethysm together with results on the representation theory of $\mathrm{Sp}(6,R)$ (Rowe *et al* 1985 and King and Wybourne 1985 (herein we follow the notation of the latter paper). She also notes briefly the existence of an 'indirect' method that exploits the complementarity that exists between $\mathrm{Sp}(6,R)$ and $\mathrm{O}(A)$. Her plethysm method seems difficult to extend to large A and does not appear to make transparent the manner in which results depend on the nucleon number A. Herein we argue that her 'indirect' method provides a more efficient and transparent method for obtaining general results. This becomes particularly the case when the 'reduced notation' associated with the A-independent representation of the $\mathrm{O}(A)\downarrow\mathrm{S}(A)$ decompositions is fully exploited.

In this paper we first give a brief review of the A-independent reduced notation and establish a table of $\mathrm{O}(A)\downarrow\mathrm{S}(A)$ decompositions. These are then used to establish the key results already alluded to.

2. Reduced notation and O(*A*) ↓ S(*A*) decompositions

The concept of reduced notation was introduced by Murnaghan (1937) and later used by Littlewood (1961) for the calculation of inner plethysms and Kronecker products for the symmetric group $\mathrm{S}(A)$. The importance of the reduced notation was emphasized by Butler and King (1973) and later applied explicitly to the development of A-independent $\mathrm{O}(A)\downarrow\mathrm{S}(A)$ decompositions (Luan Dehuai and Wybourne 1981). A concise treatment using the properties of S-function series has been given by Salam and Wybourne (1989).

The standard tensor irreducible representations $[\lambda]$ of the full orthogonal group $\mathrm{O}(A)$ may be labelled by ordered partitions (λ) of integers. The tensor irreducible representations $\{\lambda\}$ of the symmetric group $\mathrm{S}(A)$ may also be labelled by partitions (λ) but this time λ is restricted to partitions of the integer A. In reduced notation the label $\{\lambda\}=\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$ for $\mathrm{S}(A)$ is replaced by $\langle\lambda\rangle=\langle\lambda_2,\ldots,\lambda_p\rangle$. Given any irreducible representation $\langle\mu\rangle$ in reduced notation it can be converted into a standard irreducible representation of $\mathrm{S}(A)$ by prefixing it with a part $(A-|\mu|)$. For example an irreducible representation $\langle 21\rangle$ in reduced notation corresponds to $\{321\}$ in $\mathrm{S}(6)$ or $\{921\}$ in $\mathrm{S}(12)$, etc. It is just this feature that leads to the A-independent notation for $\mathrm{S}(A)$. If $A-|\mu|\geqslant\mu_1$ then the resulting irreducible representation $\{A-|\mu|,\mu\}$ is assuredly a standard irreducible representation of $\mathrm{S}(A)$. However if $A-|\mu|<\mu_1$ then the irreducible representation $\{A-|\mu|,\mu\}$ is non-standard and must be converted into standard form using Littlewood's (1950) S-function modification rules:

(1) In any S-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part increased by unity, the resulting S-function being thereby changed in sign, i.e.

$$\{\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_k\}=-\{\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_k\}\,.$$

(2) In any S-function if any part exceeds by unity the preceding part, the value of the S-function is zero, i.e.

$$\text{if}\quad \lambda_{i+1}=\lambda_i+1\qquad\text{then}\quad\{\lambda_1,\ldots,\lambda_i,\lambda_{i+1},\ldots,\lambda_k\}=0\,.$$

(3) The value of any S-function is zero if the last part is negative.

If $\mathrm{S}(A)$ is embedded in $\mathrm{O}(A)$ such that $[1] \downarrow \langle 1\rangle + \langle 0\rangle$ then in general an arbitrary irreducible representation $[\lambda]$ of $\mathrm{O}(A)$ will have a decomposition content given by (Salam and Wybourne 1989)

$$[\lambda] \downarrow \langle 1\rangle \otimes \{\lambda/G\} \tag{1}$$

where

$$G = \sum_{\epsilon} (-1)^{(e-r)/2}\{\epsilon\}$$

is an infinite S-function series with ϵ a self-conjugate partition of weight e and rank r (Black *et al* 1983). Plethysms of this type may be relatively easily evaluated to give the branching rules for $\mathrm{O}(A) \downarrow \mathrm{S}(A)$ in an A-independent manner. A list of the decompositions for all partitions (λ) of weight $\omega_\lambda \leqslant 8$ and involving not more than three parts is given in table 1. In practice the table was automatically generated by the program SCHUR. This table can be used without the need to apply modification rules for all those cases where $A - |\mu| \geqslant \mu_1$ yielding immediately the branching rules for $\mathrm{O}(A) \downarrow \mathrm{S}(A)$. Thus since

$$[3] \downarrow \langle 3\rangle + 2\langle 2\rangle + \langle 1^2\rangle + 3\langle 1\rangle + 2\langle 0\rangle$$

we have immediately for $A = 6$

$$[3] \downarrow \{3^2\} + 2\{42\} + \{41^2\} + 3\{51\} + 2\{6\}$$

and equally well for $A = 120$

$$[3] \downarrow \{117\ 3\} + 2\{118\ 2\} + \{118\ 1^2\} + 3\{119\ 1\} + 2\{120\}$$

whereas for $A = 4$ (3) leads to the non-standard $\{13\} = -\{2^2\}$ and hence

$$[3] \downarrow \{2^2\} + \{21^2\} + 3\{31\} + 2\{4\}\,.$$

If $A \geqslant 16$ there will be no entries in the table requiring use of the modification rules. It would be a comparatively easy task to extend the table to partitions of higher weight if required.

3. The fundamental Sp(2n,R) representations

There exists a true unitary infinite-dimensional representation $\tilde{\Delta}$ of the double covering group of $\mathrm{Sp}(2n, R)$, the metaplectic group $\mathrm{Mp}(2n)$, which under $\mathrm{Sp}(2n, R)$ is reducible into the sum of two irreducible representations , $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$, whose leading weights are $(\frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{3}{2}, \ldots, \frac{1}{2})$, corresponding to the highest weights $\epsilon^{1/2}\{0\}$ and $\epsilon^{1/2}\{1\}$ of the maximal compact subgroup $\mathrm{U}(n)$ (Rowe *et al* 1985, King and Wybourne 1985). The tensor powers $\tilde{\Delta}^k$ all decompose into unitary irreducible representations of $\mathrm{Sp}(2n, R)$ with the unitary irreducible representations being referred to as harmonic series representations.

Table 1. $O_n \downarrow S_n$ in reduced notation.

O_n	S_n
$[0]$	$\langle 0\rangle$
$[1]$	$\langle 1\rangle + \langle 0\rangle$
$[2]$	$\langle 2\rangle + 2\langle 1\rangle + \langle 0\rangle$
$[1^2]$	$\langle 1^2\rangle + \langle 1\rangle$
$[3]$	$\langle 3\rangle + 2\langle 2\rangle + \langle 1^2\rangle + 3\langle 1\rangle + 2\langle 0\rangle$
$[21]$	$\langle 21\rangle + 2\langle 2\rangle + 2\langle 1^2\rangle + 2\langle 1\rangle$
$[1^3]$	$\langle 1^3\rangle + \langle 1^2\rangle$
$[4]$	$\langle 4\rangle + 2\langle 3\rangle + \langle 21\rangle + 4\langle 2\rangle + 2\langle 1^2\rangle + 5\langle 1\rangle + 3\langle 0\rangle$
$[31]$	$\langle 31\rangle + 2\langle 3\rangle + 3\langle 21\rangle + 4\langle 2\rangle + \langle 1^3\rangle + 5\langle 1^2\rangle + 4\langle 1\rangle + \langle 0\rangle$
$[2^2]$	$\langle 3\rangle + \langle 2^2\rangle + 2\langle 21\rangle + 3\langle 2\rangle + \langle 1^2\rangle + \langle 1\rangle$
$[21^2]$	$\langle 21^2\rangle + 2\langle 21\rangle + \langle 2\rangle + 2\langle 1^3\rangle + 2\langle 1^2\rangle$
$[5]$	$\langle 5\rangle + 2\langle 4\rangle + \langle 31\rangle + 4\langle 3\rangle + 3\langle 21\rangle + 7\langle 2\rangle + 4\langle 1^2\rangle + 8\langle 1\rangle + 4\langle 0\rangle$
$[41]$	$\langle 41\rangle + 2\langle 4\rangle + 3\langle 31\rangle + 5\langle 3\rangle + \langle 2^2\rangle + \langle 21^2\rangle + 8\langle 21\rangle + 9\langle 2\rangle + 3\langle 1^3\rangle + 9\langle 1^2\rangle + 8\langle 1\rangle + 2\langle 0\rangle$
$[32]$	$\langle 4\rangle + \langle 32\rangle + 3\langle 31\rangle + 5\langle 3\rangle + 2\langle 2^2\rangle + \langle 21^2\rangle + 7\langle 21\rangle + 7\langle 2\rangle + 2\langle 1^3\rangle + 5\langle 1^2\rangle + 4\langle 1\rangle + \langle 0\rangle$
$[31^2]$	$\langle 31^2\rangle + 2\langle 31\rangle + \langle 3\rangle + \langle 2^2\rangle + 3\langle 21^2\rangle + 6\langle 21\rangle + 2\langle 2\rangle + \langle 1^4\rangle + 5\langle 1^3\rangle + 5\langle 1^2\rangle + \langle 1\rangle$
$[2^21]$	$\langle 31\rangle + 2\langle 3\rangle + \langle 2^21\rangle + 2\langle 2^2\rangle + 2\langle 21^2\rangle + 4\langle 21\rangle + 2\langle 2\rangle + \langle 1^3\rangle + \langle 1^2\rangle$
$[6]$	$\langle 6\rangle + 2\langle 5\rangle + \langle 41\rangle + 4\langle 4\rangle + 3\langle 31\rangle + 8\langle 3\rangle + \langle 2^2\rangle + 6\langle 21\rangle + 12\langle 2\rangle + \langle 1^3\rangle + 7\langle 1^2\rangle + 12\langle 1\rangle + 6\langle 0\rangle$
$[51]$	$\langle 51\rangle + 2\langle 5\rangle + 3\langle 41\rangle + 5\langle 4\rangle + \langle 32\rangle + \langle 31^2\rangle + 9\langle 31\rangle + 11\langle 3\rangle + 3\langle 2^2\rangle + 4\langle 21^2\rangle + 18\langle 21\rangle + 17\langle 2\rangle + 6\langle 1^3\rangle + 17\langle 1^2\rangle + 14\langle 1\rangle + 3\langle 0\rangle$
$[42]$	$\langle 5\rangle + \langle 42\rangle + 3\langle 41\rangle + 6\langle 4\rangle + 3\langle 32\rangle + \langle 31^2\rangle + 10\langle 31\rangle + 13\langle 3\rangle + \langle 2^21\rangle + 7\langle 2^2\rangle + 5\langle 21^2\rangle + 19\langle 21\rangle + 17\langle 2\rangle + \langle 1^4\rangle + 6\langle 1^3\rangle + 12\langle 1^2\rangle + 10\langle 1\rangle + 3\langle 0\rangle$
$[41^2]$	$\langle 41^2\rangle + 2\langle 41\rangle + \langle 4\rangle + \langle 32\rangle + 3\langle 31^2\rangle + 7\langle 31\rangle + 4\langle 3\rangle + \langle 2^21\rangle + 4\langle 2^2\rangle + \langle 21^3\rangle + 9\langle 21^2\rangle + 14\langle 21\rangle + 6\langle 2\rangle + 3\langle 1^4\rangle + 11\langle 1^3\rangle + 10\langle 1^2\rangle + 3\langle 1\rangle$
$[3^2]$	$\langle 41\rangle + 2\langle 4\rangle + \langle 3^2\rangle + 2\langle 32\rangle + \langle 31^2\rangle + 6\langle 31\rangle + 6\langle 3\rangle + \langle 2^2\rangle + 3\langle 21^2\rangle + 7\langle 21\rangle + 5\langle 2\rangle + 4\langle 1^3\rangle + 6\langle 1^2\rangle + 4\langle 1\rangle + \langle 0\rangle$

Table 1. (Continued)

O_n	S_n
[321]	$\langle 41\rangle + 2\langle 4\rangle + \langle 321\rangle + 3\langle 32\rangle + 3\langle 31^2\rangle + 9\langle 31\rangle + 6\langle 3\rangle + 3\langle 2^21\rangle + 7\langle 2^2\rangle + \langle 21^3\rangle + 9\langle 21^2\rangle + 15\langle 21\rangle + 6\langle 2\rangle + 2\langle 1^4\rangle + 6\langle 1^3\rangle + 6\langle 1^2\rangle + 2\langle 1\rangle$
[2^3]	$\langle 4\rangle + \langle 32\rangle + 2\langle 31\rangle + 3\langle 3\rangle + \langle 2^3\rangle + 2\langle 2^21\rangle + 3\langle 2^2\rangle + \langle 21^2\rangle + 2\langle 21\rangle + \langle 2\rangle$
[7]	$\langle 7\rangle + 2\langle 6\rangle + \langle 51\rangle + 4\langle 5\rangle + 3\langle 41\rangle + 8\langle 4\rangle + \langle 32\rangle + 7\langle 31\rangle + 14\langle 3\rangle + 2\langle 2^2\rangle + \langle 21^2\rangle + 12\langle 21\rangle + 19\langle 2\rangle + 2\langle 1^3\rangle + 12\langle 1^2\rangle + 18\langle 1\rangle + 8\langle 0\rangle$
[61]	$\langle 61\rangle + 2\langle 6\rangle + 3\langle 51\rangle + 5\langle 5\rangle + \langle 42\rangle + \langle 41^2\rangle + 9\langle 41\rangle + 12\langle 4\rangle + 4\langle 32\rangle + 4\langle 31^2\rangle + 21\langle 31\rangle + 23\langle 3\rangle + \langle 2^21\rangle + 9\langle 2^2\rangle + 10\langle 21^2\rangle + 35\langle 21\rangle + 31\langle 2\rangle + \langle 1^4\rangle$ $+13\langle 1^3\rangle + 29\langle 1^2\rangle + 23\langle 1\rangle + 6\langle 0\rangle$
[52]	$\langle 6\rangle + \langle 52\rangle + 3\langle 51\rangle + 6\langle 5\rangle + 3\langle 42\rangle + \langle 41^2\rangle + 11\langle 41\rangle + 15\langle 4\rangle + \langle 3^2\rangle + \langle 321\rangle + 10\langle 32\rangle + 6\langle 31^2\rangle + 29\langle 31\rangle + 29\langle 3\rangle + 4\langle 2^21\rangle + 17\langle 2^2\rangle + \langle 21^3\rangle + 16\langle 21^2\rangle$ $+44\langle 21\rangle + 34\langle 2\rangle + 2\langle 1^4\rangle + 14\langle 1^3\rangle + 27\langle 1^2\rangle + 20\langle 1\rangle + 5\langle 0\rangle$
[51^2]	$\langle 51^2\rangle + 2\langle 51\rangle + \langle 5\rangle + \langle 42\rangle + 3\langle 41^2\rangle + 7\langle 41\rangle + 4\langle 4\rangle + \langle 321\rangle + 5\langle 32\rangle + \langle 31^3\rangle + 10\langle 31^2\rangle + 19\langle 31\rangle + 10\langle 3\rangle + 4\langle 2^21\rangle + 11\langle 2^2\rangle + 4\langle 21^3\rangle + 22\langle 21^2\rangle + 31\langle 21\rangle$ $+13\langle 2\rangle + 7\langle 1^4\rangle + 21\langle 1^3\rangle + 20\langle 1^2\rangle + 6\langle 1\rangle$
[43]	$\langle 51\rangle + 3\langle 5\rangle + \langle 43\rangle + 3\langle 42\rangle + \langle 41^2\rangle + 9\langle 41\rangle + 12\langle 4\rangle + 2\langle 3^2\rangle + \langle 321\rangle + 8\langle 32\rangle + 6\langle 31^2\rangle + 22\langle 31\rangle + 20\langle 3\rangle + 3\langle 2^21\rangle + 10\langle 2^2\rangle + \langle 21^3\rangle + 13\langle 21^2\rangle + 28\langle 21\rangle$ $+20\langle 2\rangle + 3\langle 1^4\rangle + 12\langle 1^3\rangle + 18\langle 1^2\rangle + 13\langle 1\rangle + 4\langle 0\rangle$
[421]	$\langle 51\rangle + 2\langle 5\rangle + \langle 421\rangle + 3\langle 42\rangle + 3\langle 41^2\rangle + 10\langle 41\rangle + 9\langle 4\rangle + \langle 3^2\rangle + 4\langle 321\rangle + 13\langle 32\rangle + \langle 31^3\rangle + 13\langle 31^2\rangle + 30\langle 31\rangle + 19\langle 3\rangle + \langle 2^3\rangle + \langle 2^21^2\rangle + 12\langle 2^21\rangle + 22\langle 2^2\rangle$ $+6\langle 21^3\rangle + 29\langle 21^2\rangle + 40\langle 21\rangle + 18\langle 2\rangle + \langle 1^5\rangle + 8\langle 1^4\rangle + 18\langle 1^3\rangle + 17\langle 1^2\rangle + 7\langle 1\rangle + \langle 0\rangle$
[3^21]	$\langle 42\rangle + \langle 41^2\rangle + 5\langle 41\rangle + 4\langle 4\rangle + \langle 3^21\rangle + 2\langle 3^2\rangle + 3\langle 321\rangle + 7\langle 32\rangle + \langle 31^3\rangle + 9\langle 31^2\rangle + 16\langle 31\rangle + 7\langle 3\rangle + 4\langle 2^21\rangle + 8\langle 2^2\rangle + 3\langle 21^3\rangle + 14\langle 21^2\rangle + 17\langle 21\rangle + 6\langle 2\rangle$ $+4\langle 1^4\rangle + 10\langle 1^3\rangle + 9\langle 1^2\rangle + 3\langle 1\rangle$
[32^2]	$\langle 5\rangle + \langle 42\rangle + 3\langle 41\rangle + 5\langle 4\rangle + \langle 3^2\rangle + \langle 32^2\rangle + 3\langle 321\rangle + 8\langle 32\rangle + 4\langle 31^2\rangle + 12\langle 31\rangle + 8\langle 3\rangle + 2\langle 2^3\rangle + \langle 2^21^2\rangle + 7\langle 2^21\rangle + 10\langle 2^2\rangle + 2\langle 21^3\rangle + 8\langle 21^2\rangle + 11\langle 21\rangle$ $+5\langle 2\rangle + \langle 1^4\rangle + 2\langle 1^3\rangle + 2\langle 1^2\rangle + \langle 1\rangle$
[8]	$\langle 8\rangle + 2\langle 7\rangle + \langle 61\rangle + 4\langle 6\rangle + 3\langle 51\rangle + 8\langle 5\rangle + \langle 42\rangle + 7\langle 41\rangle + 15\langle 4\rangle + 3\langle 32\rangle + \langle 31^2\rangle + 14\langle 31\rangle + 23\langle 3\rangle + 5\langle 2^2\rangle + 3\langle 21^2\rangle + 22\langle 21\rangle + 30\langle 2\rangle + 4\langle 1^3\rangle + 19\langle 1^2\rangle$ $+26\langle 1\rangle + 11\langle 0\rangle$
[71]	$\langle 71\rangle + 2\langle 7\rangle + 3\langle 61\rangle + 5\langle 6\rangle + \langle 52\rangle + \langle 51^2\rangle + 9\langle 51\rangle + 12\langle 5\rangle + 4\langle 42\rangle + 4\langle 41^2\rangle + 22\langle 41\rangle + 25\langle 4\rangle + \langle 3^2\rangle + \langle 321\rangle + 12\langle 32\rangle + 11\langle 31^2\rangle + 45\langle 31\rangle + 43\langle 3\rangle$ $+4\langle 2^21\rangle + 19\langle 2^2\rangle + \langle 21^3\rangle + 23\langle 21^2\rangle + 65\langle 21\rangle + 52\langle 2\rangle + 3\langle 1^4\rangle + 24\langle 1^3\rangle + 49\langle 1^2\rangle + 37\langle 1\rangle + 9\langle 0\rangle$

Table 1. (Continued)

O_n	S_n
[62]	$\langle 7\rangle + \langle 62\rangle + 3\langle 61\rangle + 6\langle 6\rangle + 3\langle 52\rangle + \langle 51^2\rangle + 11\langle 51\rangle + 16\langle 5\rangle + \langle 43\rangle + \langle 421\rangle + 11\langle 42\rangle + 6\langle 41^2\rangle + 32\langle 41\rangle + 36\langle 4\rangle + 3\langle 3^2\rangle + 5\langle 321\rangle + 28\langle 32\rangle + \langle 31^3\rangle$ $+20\langle 31^2\rangle + 68\langle 31\rangle + 60\langle 3\rangle + \langle 2^3\rangle + 13\langle 2^21\rangle + 40\langle 2^2\rangle + 4\langle 21^3\rangle + 39\langle 21^2\rangle + 90\langle 21\rangle + 65\langle 2\rangle + 7\langle 1^4\rangle + 31\langle 1^3\rangle + 51\langle 1^2\rangle + 37\langle 1\rangle + 10\langle 0\rangle$
$[61^2]$	$\langle 61^2\rangle + 2\langle 61\rangle + \langle 6\rangle + \langle 52\rangle + 3\langle 51^2\rangle + 7\langle 51\rangle + 4\langle 5\rangle + \langle 421\rangle + 5\langle 42\rangle + \langle 41^3\rangle + 10\langle 41^2\rangle + 20\langle 41\rangle + 12\langle 4\rangle + \langle 3^2\rangle + 5\langle 321\rangle + 16\langle 32\rangle + 4\langle 31^3\rangle + 26\langle 31^2\rangle$ $+44\langle 31\rangle + 23\langle 3\rangle + \langle 2^21^2\rangle + 13\langle 2^21\rangle + 26\langle 2^2\rangle + 11\langle 21^3\rangle + 48\langle 21^2\rangle + 62\langle 21\rangle + 26\langle 2\rangle + \langle 1^5\rangle + 15\langle 1^4\rangle + 39\langle 1^3\rangle + 36\langle 1^2\rangle + 12\langle 1\rangle + \langle 0\rangle$
[53]	$\langle 61\rangle + 3\langle 6\rangle + \langle 53\rangle + 3\langle 52\rangle + \langle 51^2\rangle + 10\langle 51\rangle + 14\langle 5\rangle + 3\langle 43\rangle + \langle 421\rangle + 11\langle 42\rangle + 7\langle 41^2\rangle + 32\langle 41\rangle + 32\langle 4\rangle + \langle 3^21\rangle + 7\langle 3^2\rangle + 7\langle 321\rangle + 28\langle 32\rangle + \langle 31^3\rangle$ $+24\langle 31^2\rangle + 66\langle 31\rangle + 50\langle 3\rangle + \langle 2^21^2\rangle + 13\langle 2^21\rangle + 31\langle 2^2\rangle + 6\langle 21^3\rangle + 41\langle 21^2\rangle + 76\langle 21\rangle + 48\langle 2\rangle + 7\langle 1^4\rangle + 30\langle 1^3\rangle + 45\langle 1^2\rangle + 29\langle 1\rangle + 7\langle 0\rangle$
[521]	$\langle 61\rangle + 2\langle 6\rangle + \langle 521\rangle + 3\langle 52\rangle + 3\langle 51^2\rangle + 10\langle 51\rangle + 9\langle 5\rangle + \langle 43\rangle + 4\langle 421\rangle + 14\langle 42\rangle + \langle 41^3\rangle + 14\langle 41^2\rangle + 36\langle 41\rangle + 26\langle 4\rangle + \langle 3^21\rangle + 6\langle 3^2\rangle + \langle 32^2\rangle + \langle 321^2\rangle$ $+17\langle 321\rangle + 42\langle 32\rangle + 7\langle 31^3\rangle + 45\langle 31^2\rangle + 82\langle 31\rangle + 46\langle 3\rangle + 4\langle 2^3\rangle + 5\langle 2^21^2\rangle + 36\langle 2^21\rangle + 56\langle 2^2\rangle + \langle 21^4\rangle + 21\langle 21^3\rangle + 76\langle 21^2\rangle + 95\langle 21\rangle + 41\langle 2\rangle + 3\langle 1^5\rangle$ $+20\langle 1^4\rangle + 44\langle 1^3\rangle + 41\langle 1^2\rangle + 16\langle 1\rangle + 2\langle 0\rangle$
$[4^2]$	$\langle 6\rangle + \langle 52\rangle + 3\langle 51\rangle + 6\langle 5\rangle + \langle 4^2\rangle + 2\langle 43\rangle + \langle 421\rangle + 7\langle 42\rangle + 3\langle 41^2\rangle + 15\langle 41\rangle + 16\langle 4\rangle + \langle 3^2\rangle + 3\langle 321\rangle + 11\langle 32\rangle + \langle 31^3\rangle + 9\langle 31^2\rangle + 24\langle 31\rangle + 20\langle 3\rangle + \langle 2^3\rangle$ $+7\langle 2^21\rangle + 16\langle 2^2\rangle + 3\langle 21^3\rangle + 16\langle 21^2\rangle + 30\langle 21\rangle + 21\langle 2\rangle + \langle 1^5\rangle + 5\langle 1^4\rangle + 11\langle 1^3\rangle + 15\langle 1^2\rangle + 12\langle 1\rangle + 4\langle 0\rangle$
[431]	$\langle 52\rangle + \langle 51^2\rangle + 6\langle 51\rangle + 7\langle 5\rangle + \langle 431\rangle + 3\langle 43\rangle + 4\langle 421\rangle + 13\langle 42\rangle + \langle 41^3\rangle + 13\langle 41^2\rangle + 32\langle 41\rangle + 22\langle 4\rangle + 3\langle 3^21\rangle + 8\langle 3^2\rangle + \langle 32^2\rangle + \langle 321^2\rangle + 17\langle 321\rangle$ $+36\langle 32\rangle + 7\langle 31^3\rangle + 41\langle 31^2\rangle + 66\langle 31\rangle + 33\langle 3\rangle + 3\langle 2^3\rangle + 4\langle 2^21^2\rangle + 27\langle 2^21\rangle + 39\langle 2^2\rangle + \langle 21^4\rangle + 19\langle 21^3\rangle + 59\langle 21^2\rangle + 67\langle 21\rangle + 28\langle 2\rangle + 3\langle 1^5\rangle + 19\langle 1^4\rangle$ $+35\langle 1^3\rangle + 31\langle 1^2\rangle + 13\langle 1\rangle + 2\langle 0\rangle$
$[42^2]$	$\langle 6\rangle + \langle 52\rangle + 3\langle 51\rangle + 6\langle 5\rangle + \langle 43\rangle + \langle 42^2\rangle + 3\langle 421\rangle + 10\langle 42\rangle + 4\langle 41^2\rangle + 18\langle 41\rangle + 17\langle 4\rangle + \langle 3^21\rangle + 5\langle 3^2\rangle + 3\langle 32^2\rangle + \langle 321^2\rangle + 14\langle 321\rangle$ $+29\langle 32\rangle + 3\langle 31^3\rangle + 20\langle 31^2\rangle + 40\langle 31\rangle + 24\langle 3\rangle + \langle 2^31\rangle + 7\langle 2^3\rangle + 5\langle 2^21^2\rangle + 25\langle 2^21\rangle + 31\langle 2^2\rangle + \langle 21^4\rangle + 10\langle 21^3\rangle + 29\langle 21^2\rangle + 35\langle 21\rangle + 16\langle 2\rangle + 2\langle 1^5\rangle$ $+6\langle 1^4\rangle + 9\langle 1^3\rangle + 9\langle 1^2\rangle + 5\langle 1\rangle + \langle 0\rangle$
$[3^22]$	$\langle 51\rangle + 2\langle 5\rangle + \langle 43\rangle + \langle 421\rangle + 5\langle 42\rangle + 4\langle 41^2\rangle + 12\langle 41\rangle + 8\langle 4\rangle + \langle 3^22\rangle + 3\langle 3^21\rangle + 6\langle 3^2\rangle + 2\langle 32^2\rangle + \langle 321^2\rangle + 10\langle 321\rangle + 17\langle 32\rangle + 3\langle 31^3\rangle + 17\langle 31^2\rangle$ $+24\langle 31\rangle + 10\langle 3\rangle + \langle 2^3\rangle + 3\langle 2^21^2\rangle + 11\langle 2^21\rangle + 13\langle 2^2\rangle + 7\langle 21^3\rangle + 20\langle 21^2\rangle + 19\langle 21\rangle + 7\langle 2\rangle + 4\langle 1^4\rangle + 9\langle 1^3\rangle + 7\langle 1^2\rangle + 2\langle 1\rangle$

The two fundamental harmonic series unitary irreducible representations $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ will be henceforth labelled as $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$ with all those unitary irreducible representations appearing in $\tilde{\Delta}^k$ being labelled by the symbols $\langle\frac{1}{2}k(\lambda)\rangle$. Under the restriction $\mathrm{Sp}(2n, R) \downarrow \mathrm{U}(n)$

$$\tilde{\Delta}_+ = \langle\tfrac{1}{2}(0)\rangle \downarrow \epsilon^{1/2}(\{0\} + \{2\} + \{4\} + \cdots) \tag{2a}$$

$$\tilde{\Delta}_- = \langle\tfrac{1}{2}(1)\rangle \downarrow \epsilon^{1/2}(\{1\} + \{3\} + \{5\} + \cdots) \tag{2b}$$

or combining the two fundamental irreducible representations leads to the compact result

$$\tilde{\Delta}_+ + \tilde{\Delta}_- \downarrow \epsilon^{1/2} M \tag{3}$$

where

$$M = \sum_{m=0}^{\infty} \{m\}. \tag{4}$$

The harmonic series representations appearing in $\tilde{\Delta}^k$ are in a one-to-one correspondence with the terms arising in the restriction $\mathrm{Sp}(2nk, R) \downarrow \mathrm{Sp}(2n, R) \times \mathrm{O}(k)$ such that

$$\tilde{\Delta} \downarrow \sum_{\lambda} \langle\tfrac{1}{2}k(\lambda)\rangle \times [\lambda] \tag{5}$$

where the summation is over all partitions $(\lambda) = (\lambda_1, \lambda_2, \ldots)$ for which the conjugate partition $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfies the constraints

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant k \tag{6a}$$

$$\tilde{\lambda}_1 \leqslant n\,. \tag{6b}$$

This last result means that the irreducible representations of $\mathrm{O}(k)$ may be limited to partitions into at most n non-zero parts.

4. Plethysms of the fundamental irreducible representations of Sp(2n,R)

Our principal problem is to resolve the Ath power of the irreducible representations $\langle\frac{1}{2}(0)\rangle + \langle\frac{1}{2}(1)\rangle$ of $\mathrm{Sp}(2n, R)$ into its constituent irreducible representations up to a prescribed cut-off. This amounts to evaluating terms in the plethysms occurring in

$$(\langle\tfrac{1}{2}(0)\rangle + \langle\tfrac{1}{2}(1)\rangle)^A = \sum_{\nu \vdash A} f^{\nu} (\langle\tfrac{1}{2}(0)\rangle + \langle\tfrac{1}{2}(1)\rangle) \otimes \{\nu\} \tag{7}$$

where f^{ν} is the degree of the irreducible representation $\{\nu\}$ of $\mathrm{S}(A)$ and

$$(\langle\tfrac{1}{2}(0)\rangle + \langle\tfrac{1}{2}(1)\rangle) \otimes \{\nu\} = \sum_{\lambda} c^{\nu}_{\lambda} \langle\tfrac{1}{2}A(\lambda)\rangle \tag{8}$$

is a plethysm (Littlewood 1950, Wybourne 1970) and the coefficients c^ν_λ are non-negative integers. It is the evaluation of these coefficients that becomes our major task. Carvalho (1990) has tackled this problem by starting with (3) and evaluating the plethysm in the maximal compact subgroup $\mathrm{U}(n)$ and then inverting the result to obtain a list of $\mathrm{Sp}(2n, R)$ irreducible representations. All such plethysm methods rapidly lead to a combinatorial explosion in the number of terms to be considered and the results often do not have the transparency to lead to general conclusions.

Carvalho (1990) pointed out, but did not fully exploit, an alternative approach based primarily on the observation that the symmetric group $\mathrm{S}(A)$ is a subgroup of $\mathrm{O}(A)$ and hence one may consider the branching rule

$$\begin{array}{l}\mathrm{Sp}(2nA, R) \downarrow \mathrm{Sp}(2n, R) \times \mathrm{O}(A) \downarrow \mathrm{Sp}(2n, R) \times \mathrm{S}(A) \\ \tilde{\Delta} \downarrow \sum_\lambda \langle \tfrac{1}{2}A(\lambda)\rangle \times [\lambda] \downarrow \sum_{\lambda\nu} c^\nu_\lambda \langle \tfrac{1}{2}A(\lambda)\rangle \times \{\nu\}\end{array} \tag{9}$$

where the c^ν_λ are precisely the desired coefficients required in (8). Given an irreducible representation of $\mathrm{Sp}(2n, R)$, say $\langle \frac{1}{2}A(\lambda)\rangle$, one can immediately determine the representations $\{\nu\}$ of $\mathrm{S}(A)$ it is associated with, together with its multiplicity c^ν_λ, simply from a knowledge of the $\mathrm{O}(A) \downarrow \mathrm{S}(A)$ branching rule for the $[\lambda]$ irreducible representation of $\mathrm{O}(A)$.

To obtain from table 1 a list of $\mathrm{Sp}(6, R)$ irreducible representations up to excitation level $8\hbar\omega$ with a nucleon number $A \geqslant 16$ having a permutational symmetry $\{\nu\}$ with respect to S_A we proceed as follows.

(1) Convert $\{\nu\}$ into reduced notation $\langle\mu\rangle$ for S_A.
(2) Identify the first entry $\langle\mu\rangle$ in table 1 noting its multiplicity c^ν_λ and associated O_A irreducible representation $[\lambda]$.
(3) Add to the list the entry $c^\nu_\lambda\langle \frac{1}{2}A(\lambda)\rangle$.
(4) Repeat steps 2 and 3 for all entries $\langle\mu\rangle$ in the table.

For example, to obtain the list for the totally symmetric irreducible representation $\{A\}$ of S_A since in reduced notation $\{A\} \to \langle 0\rangle$ we must identify all entries $\langle 0\rangle$ in table 1. Thus $\langle 0\rangle$ occurs with multiplicity 3 for the irreducible representation [51] of O_A and hence the list will contain $3\langle \frac{1}{2}A(51)\rangle$. Putting $t = \frac{1}{2}A$ we obtain the complete list for $A \geqslant 16$ up to excitation level $8\hbar\omega$ as

$$\begin{aligned}\tilde{\Delta} \otimes \{A\} = {} & \langle t(0)\rangle + \langle t(1)\rangle + \langle t(2)\rangle + 2\langle t(3)\rangle + 3\langle t(4)\rangle + \langle t(31)\rangle + 4\langle t(5)\rangle \\ & + 2\langle t(41)\rangle + \langle t(32)\rangle + 6\langle t(6)\rangle + 3\langle t(51)\rangle + 3\langle t(42)\rangle + \langle t(3^2)\rangle \\ & + 8\langle t(7)\rangle + 6\langle t(61)\rangle + 5\langle t(52)\rangle + 4\langle t(43)\rangle + \langle t(421)\rangle \\ & + 11\langle t(8)\rangle + 9\langle t(71)\rangle + 10\langle t(62)\rangle + \langle t(61^2)\rangle + 7\langle t(53)\rangle \\ & + 2\langle t(521)\rangle + 4\langle t(4^2)\rangle + 2\langle t(431)\rangle + \langle t(42^2)\rangle .\end{aligned}$$

Since the largest partition in the above result is (8) we can conclude that the above result is valid for all $A \geqslant 16$. For values of $A < 16$ it will in some cases be necessary to make use of modification rules as shown earlier and in some such cases the $\mathrm{O}(A)$ irreducible representations $[\lambda]$ will not satisfy the constraints of (6) and must be discarded. The latter situation would arise in $A = 4$ for the $\mathrm{O}(4)$ irreducible representation $[2^2 1]$ which violates (6*a*). Our restriction to partitions (λ)

into not more than three parts ensures that for $\mathrm{Sp}(6, R)$ (6*b*) is always satisfied. Again for $A = 4$ the coefficient of $\langle 2(5)\rangle$ will be 3 and not 4 as found above since $\langle 5\rangle$ is non-standard in $\mathrm{S}(4)$ and modifies as $\{-1\ 5\} = -\{4\}$ cancelling one of the $\{4\}$ irreducible representations that comes from the reduced notation irreducible representations $\langle 0\rangle$.

5. Modification rules in reduced notation

We have noted that for small values of A recourse must be made to the S_A modification rules based upon Littlewood's S-function modification rules. In using the reduced notation it is useful to be able to write down the results directly in the reduced notation. That is, starting with a reduced notation irreducible representation $\langle\mu\rangle$ that is standard for a given S_A find all other $\langle\mu'\rangle$ that upon modification will yield $\langle\mu\rangle$, to within a sign $(\pm)$. $\langle\mu\rangle$ will be assuredly standard in S_A if $A \geqslant \mu_1 + |\mu|$. In general any reduced notation irreducible representation of the form

$$(-1)^s\langle A - |\mu| + 1, \mu_1 + 1, \mu_2 + 1, \ldots, \mu_{s-1} + 1, \mu_{s+1}, \mu_{s+2}, \ldots\rangle = \langle\mu\rangle\,. \qquad (11)$$

For example, for $A = 21$ we have the correspondence

$$\langle 4432111\rangle = \{54432111\}$$

and deduce from (11) the members of the following infinite sequence of reduced notation irreducible representations all modify to $\langle 4432111\rangle$:

$$\begin{aligned}\langle 4432111\rangle - \langle 6432111\rangle + \langle 6532111\rangle - \langle 6552111\rangle + \langle 6554111\rangle - \langle 6554311\rangle \\ + \langle 6554321\rangle - \langle 6554322\rangle + \langle 65543222\rangle - \langle 65543221\rangle + \langle 6554322211\rangle \\ - \langle 65543222111\rangle + \cdots\,.\end{aligned}$$

The above equations allow for rapid construction of results from table 1 even for values of $A < 16$. In the case of S_4 we have for $\langle 2\rangle$ the signed sequence

$$\langle 2\rangle - \langle 3\rangle + \langle 3^2\rangle - \langle 3^2 1\rangle + \langle 3^2 1^2\rangle - \cdots\,. \qquad (13)$$

To determine the number of times $\langle 2(43)\rangle$ occurs in the fourth power of the fundamental irreducible representation of $\mathrm{Sp}(6, R)$ with permutational symmetry $\{2^2\}$ we need to evaluate the number of times $\langle 2\rangle$ occurs in the decomposition of the irreducible representation $[43]$ of O_4 under the restriction $\mathrm{O}_4 \downarrow \mathrm{S}_4$. From table 1 we find that the decomposition for $[43]$ contains the terms $20\langle 2\rangle + 20\langle 3\rangle + 2\langle 3^2\rangle$. Noting (13) we can immediately conclude that $\langle 2(43)\rangle$ occurs with permutational symmetry $\{2^2\}$ twice. Continuing in that manner one finds for $A = 4$ (to weight 8)

$$\begin{aligned}\tilde{\Delta} \otimes \{2^2\} = \langle 2(2)\rangle + \langle 2(3)\rangle + 2\langle 2(21)\rangle + 2\langle 2(4)\rangle + 2\langle 2(31)\rangle + 2\langle 2(2^2)\rangle \\ + \langle 2(21^2)\rangle + 3\langle 2(5)\rangle + 4\langle 2(41)\rangle + 2\langle 2(32)\rangle + \langle 2(31^2)\rangle \\ + 4\langle 2(6)\rangle + 6\langle 2(51)\rangle + 4\langle 2(42)\rangle + 2\langle 2(41^2)\rangle + 5\langle 2(7)\rangle + 8\langle 2(61)\rangle \\ + 6\langle 2(52)\rangle + 3\langle 2(51^2)\rangle + 2\langle 2(43)\rangle + 7\langle 2(8)\rangle + 10\langle 2(71)\rangle + 8\langle 2(62)\rangle \\ + 4\langle 2(61^2)\rangle + 4\langle 2(53)\rangle + 2\langle 2(4^2)\rangle\,.\end{aligned}$$

Inspection of the above result reveals a number of interesting points. The $Sp(6, R)$ irreducible representations $\langle 2(2^21)\rangle$, $\langle 2(421)\rangle$, $\langle 2(3^21)\rangle$, $\langle 2(521)\rangle$ do not appear because firstly they involve non-standard O_4 irreducible representations that modify to zero and secondly they correspond to partitions that violate (6*a*). Three terms $\langle 2(2^3)\rangle$, $\langle 2(32^2)\rangle$, $\langle 2(42^2)\rangle$ were found to have negative coefficients. The magnitude of their coefficients were the same as those for $\langle 2(2^2)\rangle$, $\langle 2(32)\rangle$, $\langle 2(42)\rangle$, which is consistent with the O_4 modification rules for $[2^3]$, $[32^2]$, $[42^2]$, respectively, but they violate (6*a*) and hence were discarded. The O_4 modification rules can also be invoked to explain various other equalities of coefficients such as for example that between $\langle 2(6)\rangle$ and $\langle 2(61^2)\rangle$. The absence of $\langle 2(3^22)\rangle$ may at first sight seem surprising since under O_4 modification $[3^22] \to -[3^2]$; however it happens that under $O_4 \downarrow S_4$ $[3^2] \downarrow \{4\} + 2\{31\} + 2\{21^2\} + \{1^4\}$ and hence $\langle 2(3^2)\rangle$ cannot appear in $\tilde{\Delta} \otimes \{2^2\}$.

6. Concluding remarks

The above examples and comments should suffice to demonstrate the validity of the claims made in the abstract of this paper. In terms of plethysms the results obtained in this paper would be formidable and yet in terms of the reduced notation and the understanding of the $O(A) \downarrow S(A)$ branching rules the results for up to a prescribed cut-off become relatively simple and hence it is possible now to obtain a complete description of the $Sp(6, R)$ content of the nuclear symplectic model in a manner that does not depend significantly on the nucleon number A.

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